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Risk of Extreme Events in Multiobjective Decision Trees Part 2. Rare Events

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Earlier work with decision trees identified nonseparability as an obstacle to minimizing the conditional expected value, a measure of the risk of extreme events, by the well-known method of averaging out and folding back. This second of two companion papers addresses the conditional expected value that is defined as the expected outcome assuming that a random variable is observed only in the upper 100 $(1 - \alpha)$ percent of potential outcomes, where α is a cumulative probability preselected by the decision maker. An approach is proposed to overcome the need to evaluate all policies in order to identify the optimal policy. The approach is based in part on approximating the conditional expected value by using statistics of extremes. An existing convenient approximation of the conditional expected value is shown to be separable into two constituent elements of risk and can thus be optimized, along with other objectives including the unconditional expected value of the outcome, in a multiobjective decision tree. An example of sequential decision making for remediation or environmental contamination is provided. The importance of the results for risk analyis beyond the minimization of conditional expected values is pointed out.

KEY WORDS: Decision tree; multiple objectives; extreme events; conditional expected value

1. INTRODUCTION

This is the second of two companion papers on the averaging out and folding back of measures of risk of extreme events in decision trees. Decision trees have found widespread attention in literature and practice, but their use has commonly been restricted to the optimization of a single objective. Haimes *et al.* (1990) have introduced the concept of multiobjective decision trees (MODT). However, they identified a difficulty with the averaging out and folding back of conditional expected values of the outcome, a measure of the risk of extreme events. The present paper and its companion in this issue (Frohwein *et al.* 2000) are devoted to developing methods to overcome this obstacle.

Frohwein and Lambert (2000) promulgate the use of some measure of the risk of extreme events in decision analysis as a complement to the unconditional expected value of the outcome and review MODTs.

Whereas the first paper deals with the use of the conditional expected value of the outcome (a loss or damage), given that the magnitude of the outcome exceeds a given threshold β (severe events), this paper investigates an alternative measure of the risk of extreme events. This measure is the conditional expected value of the outcome, *given that the magnitude of the outcome falls in the upper 100* $(1 - \alpha)$ *percent tail of the cumulative probability distribution of outcomes.* This defines, on the basis of cumulative probability α, a range of rare events that are known to be of concern to the decision maker. This type of conditional expected value can also be interpreted as the "worst 100 $(1 - \alpha)$-

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in-100 expectation." Calculating conditional expected values on the basis of a fixed probability α may be less intuitive than calculating such values on the basis of fixed outcome threshold β (cf. Frohwein and Lambert [2000]). However, it should be noted that agencies often regulate on the basis of an extreme percentile $F^{-1}(\alpha)$ (e.g., 95th percentile) rather than the probability of exceeding some fixed outcome threshold. Therefore, it is plausible to also consider the conditional expected value on the basis of a fixed probability threshold α. It is not claimed that this measure of the risk of extreme events can, by itself, capture all facets of risk—no measure can. However, the conditional expected value under consideration here can, possibly in conjunction with other measures of risk, provide helpful information to the decision maker. For example, a manager may be concerned about the expected performance of his worst of 10 employees ($\alpha = 0.9$), or an environmental scientist about the expected contamination measured in the worst of 100 soil samples ($\alpha = 0.99$). The companion paper (Frohwein and Lambert 2000) provides references on the use of conditional expected values as a measure of the risk of extreme events.

The following section discusses the problems with averaging out and folding back conditional expected values, defined by a fixed nonexceedance probability α, in decision trees. Next, the developments to overcome these difficulties are outlined. As required by the proposed approach, approximate expressions for the conditional expected values are then derived and conditions that enable the sequential optimization of the conditional expected value are established. Then, the optimization process is summarized and depicted in a flowchart. After the key results have been reiterated, an example (contamination remediation) is provided to illustrate the application of the proposed method. Finally, concluding remarks highlight the general importance of the results for risk analysis.

2. PROBLEMS WITH AVERAGING OUT AND FOLDING BACK CONDITIONAL EXPECTED VALUES IN DECISION TREES

The conditional expected value of the outcome, conditioned on the outcome magnitude falling in the upper 100 $(1 - \alpha)$ percent of possible outcome magnitudes, as a function of the chosen policy s, can be expressed as

$$f_{4,\alpha}(s) = E[X \mid X \geq F^{-1}(\alpha; s)], \tag{1}$$

where X is a random variable, $F^{-1}(\cdot; s)$ denotes the inverse of the cumulative probability distribution of X, given policy s, and α is the decision maker's nonexceedance probability of concern. The notation "$f_{4,\alpha}$" for the conditional expected value follows previous papers on the topic of conditional expected values as measure of the risk of extreme events (Asbeck and Haimes 1984, Haimes *et al.* 1990). Averaging out and folding back the conditional expected values $f_{4,\alpha}$ cannot be accomplished in the same manner as for unconditional expected values, i.e.,

$$E[X \mid X > F^{-1}(\alpha)] \neq p_1\, E[X_1 \mid X_1 > F^{-1}(\alpha)] + \ldots + p_n\, E[X_n \mid X_n > F^{-1}(\alpha)], \tag{2}$$

where p_i denotes the probability of obtaining random variable X_i and where the p_i's sum to 1.

Haimes *et al.* (1990) first identified this difficulty, which is ascribed to the nonseparability and nonmonotonicity of conditional expected values. Unconditional expected values, on the other hand, are separable and monotonic. For a mathematical definition of separability and monotonicity see, e.g., Li (1990).

Frohwein and Lambert (2000) show that the conditional expected value $f_{4,\beta}$, conditioned on the outcome exceeding threshold β, is second-order separable (Li 1990, Li and Haimes 1990, 1991) because it can be expressed and optimized in terms of the partial expected value $f_{4,\beta}^*$ and the exceedance probability $\phi_{4,\beta}$. However, this approach cannot be used for the conditional expected value $f_{4,\alpha}$ because it is infeasible to define an $f_{4,\alpha}^* \equiv (1 - \alpha)\, E[X \mid X \geq F^{-1}(\alpha)]$. To do so would require replacing the outcome threshold β in the expressions for $f_{4,\beta}^*$ by the threshold $F^{-1}(\alpha)$ in the expressions for $f_{4,\alpha}^*$ (recall that the decision maker now is assumed to be concerned about outcome magnitudes that fall in the upper 100 $(1 - \alpha)$% of all outcomes). However, here $F^{-1}(\alpha)$ denotes the inverse probability distribution of outcomes associated with some policy s *at the root node* of the decision tree. Because $F^{-1}(\alpha)$ cannot be known before the root node is reached, $f_{4,\alpha}^*$ cannot be determined for the *terminal chance nodes* of the decision tree, so that the averaging-out-and-folding-back technique fails (in the absence of some other, k-th order separable decomposition of $f_{4,\alpha}$, whose existence or nonexistence remains to be established).

3. OVERVIEW OF THE NEW FOLDING BACK OF CONDITIONAL EXPECTED VALUES

The insights from the previous section motivate the development of a different method for eliminating at intermediate nodes of the decision tree at least some policies that are inferior with respect to the risk of rare events, as measured by the conditional expected value defined by a fixed nonexceedance probability α.

An overview of the subsequent sections is as follows. It is assumed that the decision maker is concerned with events that are represented in the right tails of the probability distributions of outcomes of alternative policies (at the root node of the decision tree). Following others (e.g., Castillo 1988), it is proposed to replace the tail of the probability distribution of the outcomes of each overall policy at the root node of the decision tree by the tail of one of three limiting distributions. Next, it is convenient to adopt existing approximations of the conditional expected value $f_{4,\alpha}$ in terms of two extremal distribution parameters. Using these expressions, $f_{4,\alpha}$ can eventually be approximated as a function of the partitioning probability α, the probabilities $q_1 = F(x_1)$ and $q_2 = F(x_2)$ of not exceeding some damages x_1 and x_2, respectively, as well as an upper bound λ of outcome, where appropriate. Whereas α, x_1, and x_2 (and λ) are fixed parameters, the cumulative probabilities q_1 and q_2 are assessed for each terminal chance node, i.e., they are dependent on the choice of policy. The averaging-out-and-folding-back approach can be used on them in a multiobjective decision tree. Conditions are established under which the derived approximation of the conditional expected value $f_{4,\alpha}$ is strictly increasing or decreasing in both q_1 and q_2. Under these conditions, the approximation of $f_{4,\alpha}$ is "second-order separable," i.e., it is a strictly increasing or decreasing function of two measures of performance (q_1 and q_2), the measures of performance themselves being expected values and thus separable and monotonic and lending themselves to averaging out and folding back (cf. Proposition 1 in Frohwein and Lambert [2000]). The policy with the minimal value of $f_{4,\alpha}$ will be found among the efficient policies with respect to min $(q_1, -q_2)$ at the root node of the decision tree (cf. Proposition 2 in Frohwein and Lambert 2000). Thus, inferior policies (with respect to min $[q_1, -q_2]$) can be identified and eliminated at intermediate nodes, thereby avoiding averaging out and folding back of all policies to the root node of the tree.

4. APPROXIMATING THE CONDITIONAL EXPECTED VALUE

Following extreme-value theory, three possible limiting distributions for the right tails of the probability distributions are defined as follows (Castillo 1988):

$$F_{\text{Gumbel}}(x) = \exp\left(-e^{-k(x-v)}\right), \tag{3}$$

where v is the location parameter and k is the scale parameter;

$$F_{\text{Frechet}}(x) = \exp\left(-\left(\frac{v}{x-\lambda}\right)^k\right),\ x > \lambda,\quad F(x) = 0 \text{ otherwise}, \tag{4}$$

where λ is the lower bound of the distribution (location parameter), v is the scale parameter, and k is the shape parameter;

$$F_{\text{Weibull}}(x) = \exp\left(-\left(\frac{\lambda - x}{v}\right)^k\right),\ x \le \lambda,\quad F(x) = 1 \text{ otherwise}, \tag{5}$$

where λ is the upper bound of the distribution (location parameter), v is the scale parameter, and k is the shape parameter.

According to Ang and Tang (1984), the Gumbel (Gumbel Type I) distribution can be used for approximating a distribution with an exponentially decaying tail and no upper bound, the Frechet (Gumbel Type II) distribution for a distribution with a polynomially decaying tail and no upper bound, and the Weibull (Gumbel Type III) distribution for a distribution with an upper bound of λ. Although not all distributions converge to one of the three Gumbel Types, they are appropriate for many engineering applications (Ang and Tang 1984, Castillo 1988).

Mitsiopoulos *et al.* (1991) have shown that for the three limiting tail forms under consideration, the conditional expected value $f_{4,\alpha}$ can be approximated as a function of the extremal parameters u_α and δ_α (see Table I). The "characteristic largest value" u_α is defined as

$$u_\alpha = F^{-1}(\alpha), \tag{6}$$

and the "inverse measure of dispersion" δ_α is defined as

$$\frac{1}{\delta_\alpha} = \frac{du_\alpha}{d\ln(n)}, \tag{7}$$

Table I. Conditional Expected Value for Rare Events (Mitsiopoulos *et al.* 1991)

Tail equivalence	$f_{4,\alpha} = E[X \mid X \ge F^{-1}(\alpha)]$
Gumbel (Type I)	$u_\alpha + \frac{1}{\delta_\alpha}$
Frechet (Type II)	$u_\alpha + \frac{1}{\delta_\alpha} + \frac{1}{(\delta_\alpha)^2\left(u_\alpha - \frac{1}{\delta_\alpha}\right)}$
Weibull (Type III)	$u_\alpha + \frac{1}{\delta_\alpha} - \frac{1}{(\delta_\alpha)((\lambda - u_\alpha)\delta_\alpha + 1)}$

where the definition $n = 1/(1 - \alpha)$ is adopted. Mitsiopoulos *et al.* (1991) also investigated the errors introduced by the approximations and found them to be quite small for the examples that they examined. For $\alpha = 0.98$, the error was found to be no larger than 3.3%. Furthermore, the error decreased with increasing α for all investigated cases.

Once an appropriate limiting form (Gumbel, Frechet, or Weibull) for the tail of the original distribution has been assumed, the extremal parameters u_α and δ_α and hence $f_{4,\alpha}$ can be expressed as functions of two cumulative probabilities q_1 and q_2, which are associated with two damage levels x_1 and x_2. (Pratt *et al.* [1995] also use two fractiles to estimate the parameters of some two-parameter distributions.) To do so, q_1 and q_2 are equated with $F_{Limit}(x_1)$ and $F_{Limit}(x_2)$, respectively, where Limit = {Gumbel, Frechet, Weibull}. It is assumed that, for the Frechet distribution, the lower bound on the outcome is zero for all policies and that, for the Weibull distribution, the upper bound is the same and known for all policies.

Table II (Frohwein 1999) provides the resulting approximate expressions for the conditional expected value $f_{4,\alpha}$ that are functions of q_1 and q_2 (assessed for each terminal chance node and to be optimized—to minimize $f_{4,\alpha}$—at the root node by choosing an appropriate policy), and that have the parameters x_1, x_2, α, and possibly λ; these are fixed. The major advantage provided by the expressions in Table II is that, unlike $f_{4,\alpha}$ or any of the distribution parameters, the cumulative probabilities q_1 and q_2, by virtue of their separability and monotonicity as expected values, can be averaged out and folded back to the root node of the decision tree for any given policy (cf. Proposition 1 in Frohwein and Lambert 2000). With the assumption of a Gumbel Type for the tail, $f_{4,\alpha}$ is then obtained at the root node by using the results in Table II. Consistent with Mitsiopoulos *et al.* (1991), where the product $u_\alpha\ \delta_\alpha$ has to be larger than 1 for the conditional expected value $f_{4,\alpha}$ to be positive for the Frechet tail, it will be required here that $k > 1$ for that particular tail type.

Note that although the value of α is chosen on the basis of the decision maker's concerns (recall that the decision maker is assumed to be interested in the "upper 100 $(1 - \alpha)$ percent of outcomes"), the pairs (x_1, q_1) and (x_2, q_2) merely serve to calculate the conditional expected value. The appropriate selection of x_1 and x_2, $x_1 < x_2$, is driven by the requirement that the pairs (x_1, q_1) and (x_2, q_2) lie in the probability distribution tails and by further conditions to be established below (cf. Corollary 1).

Table II. Conditional Expected Value in Terms of Two Fractiles and the Decision Maker's Partitioning Probability $\alpha = 1 - 1/n$ (Frohwein 2000)

Assumption of distribution tail	$f_{4,\alpha} = E[X \mid X \geq F^{-1}(\alpha)]$
Gumbel (Type I)	$x_2 + \dfrac{\ln(-\ln(q_2)) - \ln(1-\alpha) + 1}{\ln(-\ln(q_2)) - \ln(-\ln(q_1))} \cdot (x_1 - x_2)$
Frechet (Type II)	
$k = \dfrac{\ln\left(\dfrac{\ln(q_2)}{\ln(q_1)}\right)}{\ln\left(\dfrac{x_1}{x_2}\right)} > 1$	$x_1(-\ln(q_1) \cdot n)^{1/k}\left(\dfrac{1}{1 - \dfrac{1}{k}}\right)$
Weibull (Type III)	
$k = \dfrac{\ln\left(\dfrac{\ln(q_2)}{\ln(q_1)}\right)}{\ln\left(\dfrac{\lambda - x_2}{\lambda - x_1}\right)} > 0$	$\lambda - (\lambda - x_1)\left(\dfrac{1}{-\ln(q_1)\ ?\ n}\right)^{1/k}\left(\dfrac{1}{1 + \dfrac{1}{k}}\right)$

5. SEQUENTIAL OPTIMIZATION OF THE CONDITIONAL EXPECTED VALUE

It is unclear *a priori* whether the conditional expected value $f_{4,\alpha}$ is a strictly increasing or decreasing function of q_1 and q_2. Such clear behavior would be useful for a sequential elimination process of inferior policies at intermediate nodes in the decision tree. If $f_{4,\alpha}$ were strictly increasing in q_1 and strictly decreasing in q_2, for example, policy alternatives not efficient with respect to the multiobjective optimization min $(q_1, -q_2)$ could be eliminated at intermediate nodes of the decision tree as k-th order separability is invoked (Geoffrion 1967, Haimes *et al.* 1990, Li 1990, Li and Haimes 1990, 1991, Frohwein and Lambert 2000). To establish conditions for strictly increasing or decreasing behavior of $f_{4,\alpha}$, the partial derivatives $\partial f_{4,\alpha}/\partial q_1$ and $\partial f_{4,\alpha}/\partial q_2$ are determined on the basis of Table II (Frohwein 2000).

Proposition 1. For each of the three limiting distributions considered in this paper (Gumbel, Frechet, or Weibull), it holds that $\partial f_{4,\alpha}/\partial q_1 \geq 0$ and $\partial f_{4,\alpha}/\partial q_2 \leq 0$ if α is not smaller than $\alpha_{crit.}$ as defined in Table III.

Table III shows that, depending on the limiting distribution, $\alpha_{crit.}$ is a function of q_2 alone (Gumbel), or $\alpha_{crit.}$ is a function of q_1 and q_2 and has parameters x_1 and x_2 (Frechet, Weibull) and λ (Weibull). (Recall that x_1, x_2, and λ are fixed parameters, in contrast to q_1

Table III. Policy Elimination at Intermediate Nodes for Decision Maker's Partitioning Probability $\alpha \geq \alpha_{crit.}$ (Frohwein 2000)

Assumption of distribution tail	$\alpha_{crit.}$
Gumbel (Type I)	$1 + \exp(1) \ln q_2$
Frechet (Type II) $k = \dfrac{\ln\left(\dfrac{\ln(q_2)}{\ln(q_1)}\right)}{\ln\left(\dfrac{x_1}{x_2}\right)} > 1$	$1 + \exp\left(\dfrac{1}{1-\dfrac{1}{k}}\right)\ln(q_2)$
Weibull (Type III) $k = \dfrac{\ln\left(\dfrac{\ln(q_2)}{\ln(q_1)}\right)}{\ln\left(\dfrac{\lambda - x_2}{\lambda - x_1}\right)} > 0$	$1 + \exp\left(\dfrac{1}{1+\dfrac{1}{k}}\right)\ln(q_2)$

and q_2, which are influenced by the choice of policy.) For the derivations that lead to Proposition 1 and the results in Table III, see Frohwein (2000). Further note that $\alpha_{crit.}$ is bounded from above by q_2, regardless of the tail type.

Corollary 1 follows directly from Proposition 1 and Geoffrion (1967) as well as Li (1990) and Li and Haimes (1990, 1991) (see Proposition 2 in Frohwein and Lambert 2000).

Corollary 1. For $\alpha \geq \alpha_{crit.}$ for all policies s, the policy with the minimal conditional expected value $f_{4,\alpha}$ is found at the root node of the decision tree among the efficient policies with respect to the multiobjective minimization of the cumulative probability $q_1(s) = F(x_1; s)$ and maximization of the cumulative probability $q_2(s) = F(x_2; s)$ [or minimization of $-q_2(s) = -F(x_2; s)$].

Note that the same tail form (Gumbel, Frechet, or Weibull) has to be assumed for all efficient policies and that the value of $\alpha_{crit.}$ changes with the assumed tail form (Table III). The damage levels x_1 and x_2 must be chosen such that $\alpha \geq \alpha_{crit.}$ for all policies s at the root node of the decision tree.

Figure 1 interprets Table III graphically for the Weibull tail with contour plots for $\alpha_{crit.}$ as a function of q_1 and q_2 (with parameters $x_1 = 100$, $x_2 = 110$, and $\lambda = 120$). The notation "0.9_i" denotes i 9's to the right of the decimal point (e.g., $0.9_4 = 0.9999$). As an example, for values of q_1 and q_2 from the highlighted area in the contour plot, $f_{4,\alpha} = 0.99$ decreases with decreasing q_1 and increasing q_2 for the assumed Weibull tail.

The relation between x_1, x_2, q_1, q_2, and α for two different policies is depicted in Fig. 2. Note also from the figure how the choice of larger or smaller q_1, q_2 with fixed x_1, x_2 would affect the weight of the distri-

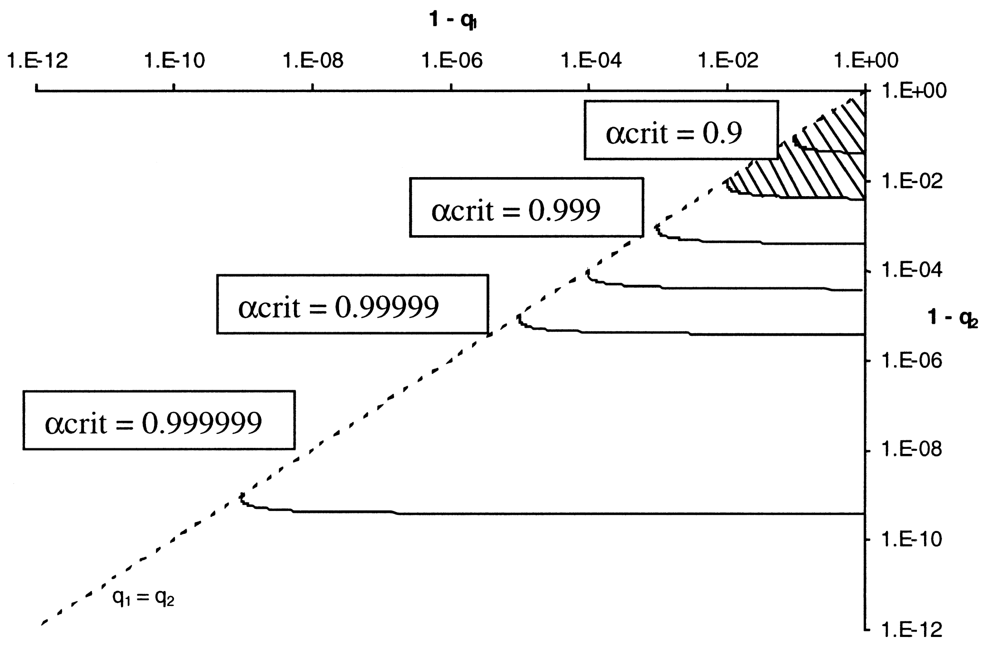

Fig. 1. Decision maker's $\alpha_{crit.}$ as a function of q_1 and q_2 for $x_1 = 100$, $x_2 = 110$, $\lambda = 120$, for an assumption of the Weibull tail.

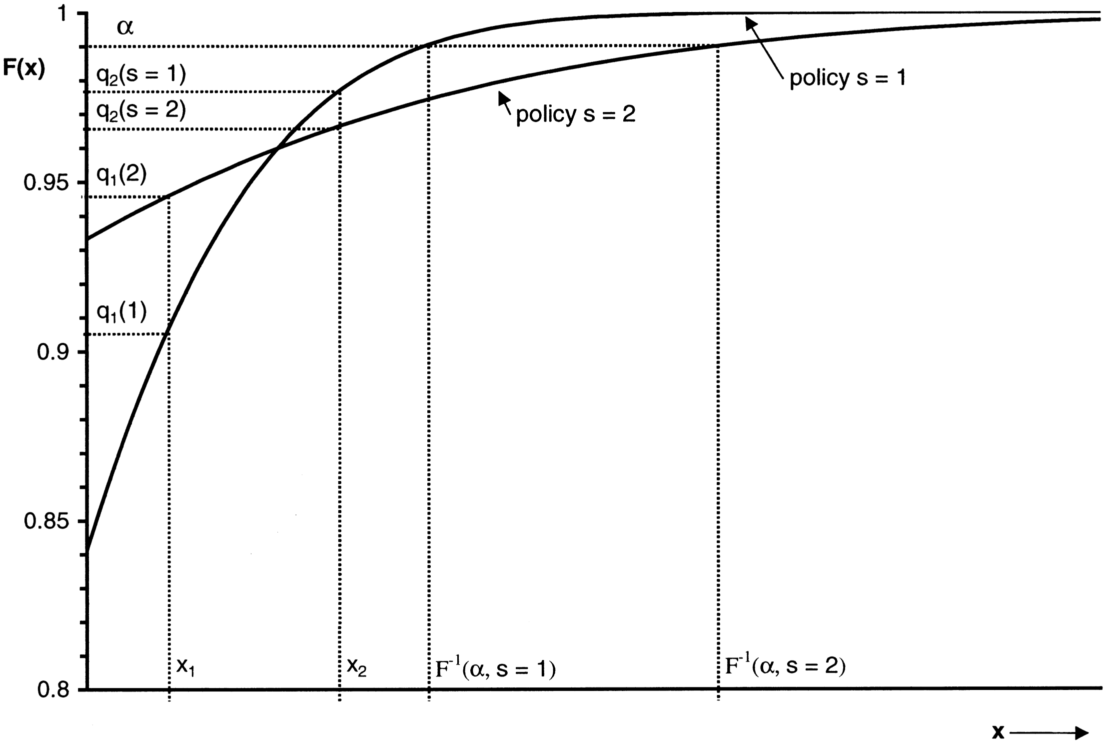

Fig. 2. Policy $s = 1$ is preferred to policy $s = 2$ because $q_1(s = 1) < q_1(s = 2)$ and $q_2(s = 1) > q_2(s = 2)$.

bution tail and thus the magnitude of the conditional expectation $f_{4,\alpha}$.

The approach presented here for minimizing $f_{4,\alpha}$ is wholly compatible with the addition of other objectives, such as minimizing cost or the unconditional expected value of the outcome. The policies that are *efficient* with respect to minimizing $f_{4,\alpha}$ and optimizing the additional objectives will be found in the expanded set of policies that are efficient with respect to min $(q_1, -q_2)$ and the optimization of the additional objectives (see Proposition 3 in Frohwein and Lambert 2000).

6. OPTIMIZATION PROCEDURE

The approach introduced in this paper is illustrated in a simplified optimization process flowchart in Fig. 3. The dashed-line box indicates the step that is different from (i.e., additional to) the optimization approach for $f_{4,\beta}$ (cf. Frohwein and Lambert 2000). Before the multiobjective optimization is performed by using the decision tree, an approximation of the conditional expected value $f_{4,\alpha}$ that is second-order separable has to be found. This approximation is a function of two cumulative probabilities, q_1 and q_2, that are then averaged out and folded back in the multiobjective decision tree (rather than $f_{4,\beta}$* and $\phi_{4,\beta}$ in the case of the conditional expected value $f_{4,\beta}$). The remainder of the process is comparable to that described for $f_{4,\beta}$ (Frohwein and Lambert 2000).

7. OVERVIEW OF KEY RESULTS

Following are key implications and results of the preceding sections:

- Based on previous approximations of the conditional expected value $f_{4,\alpha}$ in terms of two ex-

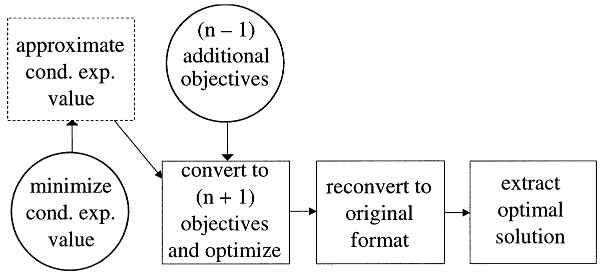

Fig. 3. Flowchart of optimization process for $f_{4,\alpha}$.

tremal distribution parameters (Mitsiopoulos *et al.* 1991), an expression for $f_{4,\alpha} = f_{4,\alpha}$ (q_1, q_2; x_1, x_2, λ) can be derived for the policies at the root node of the decision tree by assuming the right tail of either a Gumbel, Frechet, or Weibull distribution (Table II).

- The approximation of the conditional expected value $f_{4,\alpha}$ relies on statistical estimation of the cumulative probabilities q_1 and q_2 for the terminal chance nodes and on preselection of the parameters x_1, x_2, $\lambda_{Weibull}$, and α.
- When $\alpha \geq \max_s \alpha_{crit.}(q_1(s), q_2(s); x_1, x_2, \lambda_{Weibull})$ (see Table III) holds for the policies s at the root node of the decision tree, policies that are not efficient with respect to min (q_1, $-q_2$) can be eliminated at intermediate nodes of the decision tree (Corollary 1).
- A convenient upper bound on $\alpha_{crit.}$ is $\max_s q_2(s)$.

8. EXAMPLE

Consider the problem of mitigating soil contamination at a site as represented by the decision tree in Fig. 4. In this example, attention will be focused on the minimization of $f_{4,\alpha}$, with the understanding that in practice other objectives (e.g., unconditional expected damage) could be added straightforwardly to the analysis in an MODT (Haimes *et al.* 1990).

In this example, the decision maker is concerned with the worst 1-in-100 average residual soil contamination (in parts per billion, ppb) measured after completion of the mitigation efforts, i.e., $\alpha = 0.99$. The first stage of decision making mandates a choice between two alternative sampling plans for site characterization. Using the chosen sampling plan, the level of dispersion (high/low) will then be determined; the probabilities of identifying a high or low level of dispersion are dependent on the sampling plan. Once the contamination level is known, the second phase of decision making must choose an appropriate remediation of the contaminated site (e.g., soil excavation vs. pump and treat). Finally, after completion of the treatment, the residual contamination level (in ppb) will be measured. The cumulative probabilities q_1 and q_2 for the residual contamination levels $x_1 = 100$ ppb and $x_2 = 110$ ppb upon completion of mitigation, as a function of the chosen sampling plan, treatment option and the dispersion level, have been assessed by a scientist and statistician and are noted at the terminal chance nodes in the format $\{q_1, q_2\}$ in Fig. 4. Single-objective optimizations ("max q_2" and "min q_1") using the decision tree reveal that no *overall* policy s has a value of $q_2(s)$ larger than 0.9872 or a value of $q_1(s)$ smaller than 0.9035. Consequently, the choice of x_1 and x_2 is indeed feasible for the elimination of (sub) policies at intermediate nodes, given $\alpha = 0.99$, because $\alpha_{crit.} < \max_s q_2(s) = 0.9872 < \alpha = 0.99$ and $\min_s q_1(s) > 0.9$ (i.e., assumption of events from the distribution tails holds).

The $\{q_1, q_2\}$ vectors are averaged out and folded back through the decision tree according to the optimization strategy min (q_1, $-q_2$). At the intermediate nodes and the root node in Fig. 4, only efficient policies are indicated. In this example, four out of eight possible overall policies are efficient at the root node; they are policies (1.1, (2.1, 2.3)), (1.2, (2.5, 2.8)), (1.2, (2.6, 2.7)), and (1.2, (2.6, 2.8)), the numbers referring to the selected chance nodes at both stages of decision making.

On the basis of Corollary 1, the policy with the minimum approximated $f_{4,\alpha}$ value is known to be among the four efficient policies. The approximated value of $f_{4,\alpha}$ can be calculated with the formula for the appropriate tail distribution from Table II.

Consider that the analyst comes to the conclusion that none of the efficient policies at the root node has a probability distribution with an upper bound; however, whether a Gumbel or a Frechet tail would be more appropriate for the non-dominated alternatives is unclear. Because the conditions of Corollary 1 hold for both tail types, the approximated conditional expected value $f_{4,\alpha}$ can be calculated for the assumptions of both Frechet and Gumbel tails. The sensitivity of the results (i.e., the identity of the policy with the lowest value of $f_{4,\alpha}$ and the value of the minimal $f_{4,\alpha}$) to different assumptions of tail distributions can then be examined. In this particular case, it is found that the identity ((1.1, (2.1, 2.3))) of the policy with the lowest value of $f_{4,\alpha}$ does not change, regardless of the two approximation types used. However, the assumption of a Frechet-type tail entails a slightly larger minimum value of $f_{4,\alpha}$ (127.2 ppb) than the Gumbel-type tail (124.6 ppb) (see Table IV).

Figure 5 illustrates the progress of the sequential eliminations associated with the averaging-out-and-folding-back procedure by indicating at which stage of the optimization inferior alternatives are excluded from further consideration. At the first stage of folding back (decision nodes A through D), two out of eight policies—those that include chance node "2.2" as a potential outcome—can be eliminated. At the next averaging-out stage (chance nodes "1.1," "1.2"), one more policy is excluded from further consideration. Finally, at the root node R, one more policy is identified as inferior. Only four out of the original

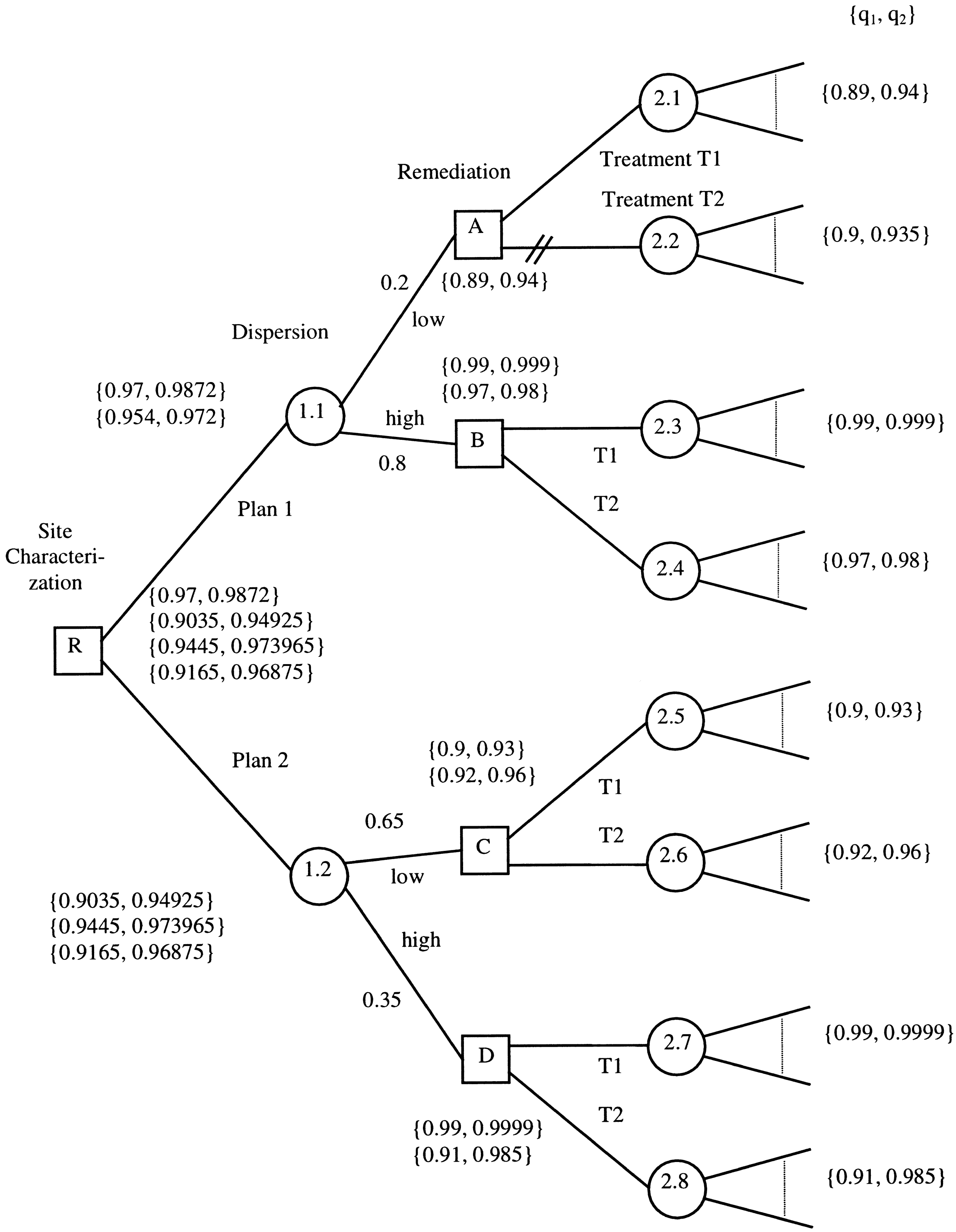

Fig. 4. Decision tree where $\{q_1, q_2\}$ represent the cumulative probabilities of 100 ppb and 110 ppb, respectively, of post-remediation soil contamination.

eight policies remain (highlighted in Fig. 5 by a connecting line) that are efficient in terms of min $(q_1, -q_2)$ and that need to be screened for the minimal value of $f_{4,\alpha}$, as described previously.

The example has shown that not all remediation policies need to be evaluated. The risk of extreme residual contamination has been handled as an optimal balance of two extreme probabilities of exceedance.

Table IV. Values of $f_{4,\alpha}$ for Efficient Policies in Example—Sensitivity Analysis

	$f_{4,\alpha}$ (in pbb)	
Policy	Gumbel	Frechet
(1.1, (2.1, 2.3))	**124.6**	**127.2**
(1.2, (2.5, 2.8))	149.7	162.5
(1.2, (2.6, 2.7))	135.5	141.5
(1.2, (2.6, 2.8))	131.1	135.4

Note: Bold indicates policy with the lowest value of $f_{4,\alpha}$

9. CONCLUDING REMARKS

In this paper, an approach has been presented for sequential decision making when one of the objectives is minimizing the risk of rare events. Its novelty is that, under the assumption of a limiting form of the tails of the probability distributions of the policies at the root node, it enables the use of the well-known averaging-out-and-folding-back procedure in order to sequentially eliminate some subpolicies at intermediate nodes of the decision tree that are inferior with respect to the risk of rare events, as measured by $f_{4,\alpha}$. Similar to the minimization of the conditional expected value $f_{4,\beta}$ (Frohwein and Lambert 2000), the risk of rare events is separated into two constituent elements of risk, in the present case "q_1" and "$-q_2$." The concept of multiobjective decision trees is then used for simultaneously minimizing these two substitute objectives (additional objectives may be considered).

The companion paper (Frohwein and Lambert 2000) addressed the feasibility of combining the approach to optimizing the risk of severe events with other techniques such as screening methods to thin the set of efficient subpolicies at intermediate nodes. These remarks apply to the present approach as well.

The approaches for sequential optimization of the risk of rare and severe events presented in this and the companion paper constitute both an extension and an application of the concept of multiobjective decision trees (Haimes *et al.* 1990) and of previous results in decision analysis with attention to extreme events. It is interesting to note that the approaches taken to decompose the conditional expected values $f_{4,\alpha}$ and $f_{4,\beta}$ (Frohwein and Lambert 2000) are different (approximation and use of two cumulative probabilities q_1 and q_2, and use of partial expected value $f_{4,\beta}^*$ and exceedance probability $\phi_{4,\beta}$, respectively). At the same time, the underlying approach to optimizing $f_{4,\alpha}$ and $f_{4,\beta}$ is the same—use of second-order separability and multiobjective decision

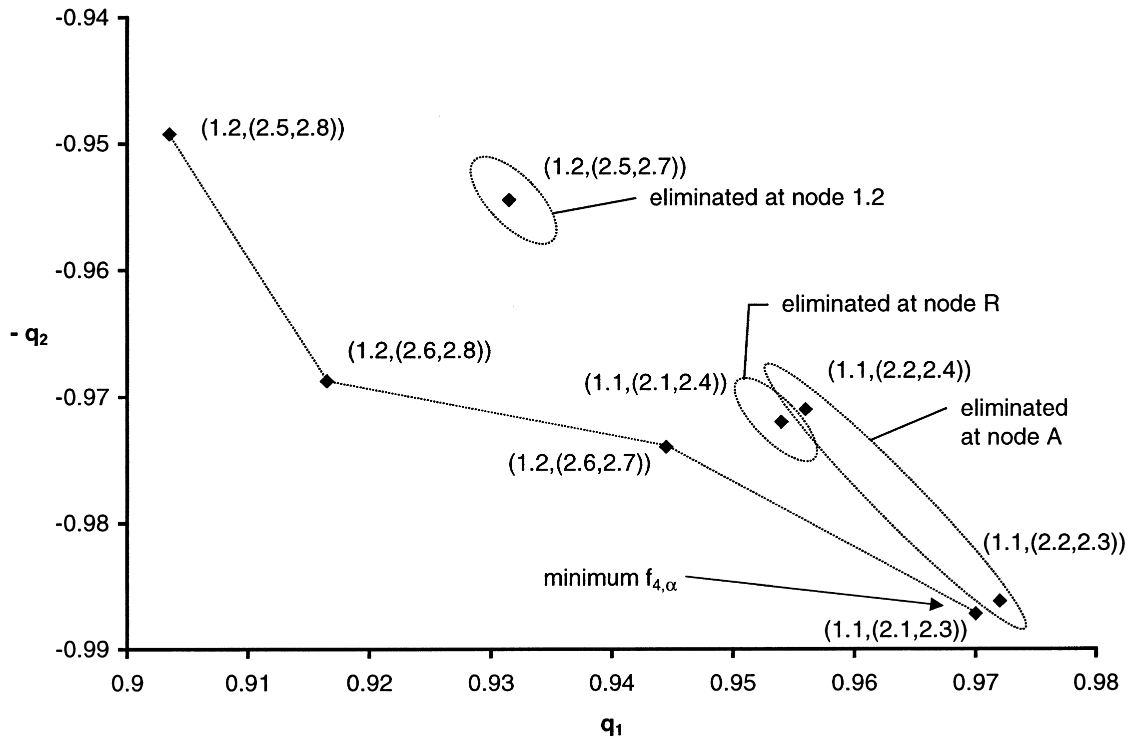

Fig. 5. Elimination of inferior policies where the policy with the minimum $f_{4,\alpha}$ (in ppb) is highlighted.

trees. This proves the flexibility of the approach that has been presented in the two papers—being applicable to various nonseparable measures of the risk of extreme events—and emphasizes its importance for risk analysis beyond the optimization of conditional expected values. It is thought that these new developments can advance the practice of incorporating measures of risk of extreme events in its different facets into multistage decision analysis and decision making.

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APPENDIX: NOTATION

α Decision maker's nonexceedance probability (cumulative probability) of concern

β Decision maker's outcome threshold of concern

$\alpha_{crit.}$ $\partial f_{4,\alpha}/\partial q_1 \geq 0$ and $\partial f_{4,\alpha}/\partial q_2 \leq 0$ for decision maker's $\alpha \geq \alpha_{crit.}$

∂_α Inverse measure of dispersion

$\phi_{4,\beta}$ Probability of outcome X at least attaining threshold β, $\phi_{4,\beta} \equiv P(X \geq \beta)$

λ Distribution parameter (lower/upper bound)

$\partial\cdot/\partial\cdot$ Partial derivative

$E[\cdot]$ Expected value

$E[\cdot|\text{condition}]$ Conditional expected value, given some condition

$\exp(\cdot)$ exponential function

$F^{-1}(\cdot)$ Inverse cumulative probability distribution function

$F^{-1}(\cdot; s)$ Inverse cumulative probability distribution function, given policy s

$f_{4,\alpha}$ Conditional expected value of outcome X, given that the magnitude of the outcome falls in the upper 100 $(1 - \alpha)$ percent tail of the cumulative probability distribution of outcomes, $f_{4,\alpha} \equiv E[X \mid X \geq F^{-1}(\alpha)]$

$f_{4,\beta}$ Conditional expected value of outcome X, given that the magnitude of the outcome attains at least the threshold β, $f_{4,\beta} \equiv E[X \mid X \geq \beta]$

$f_{4,\beta}^*$ partial expected value of outcome X, given that the magnitude of the outcome attains at least the threshold β, $f_{4,\beta}^* = \phi_{4,\beta} \cdot$

i Index

k Distribution parameter

$\ln(\cdot)$ Natural logarithm

max Maximize

$\max_i$ {argument} Select i such that the argument, which changes with i, is maximized

min Minimize

$\min_i$ {argument} Select i such that the argument, which changes with i, is minimized

$\min(\cdot, \cdot)$ Multiobjective minimization with respect to two objectives

n $n \equiv 1 / (1 - \alpha)$

q_1, q_2 nonexceedance (cumulative) probabilities in the tail of the outcome probability distribution

$q_1(s), q_2(s)$ Nonexceedance (cumulative) probabilities in the tail of the outcome probability distribution of some policy s

s Policy (at root node of decision tree)

u_α Characteristic largest value

v Distribution parameter

x_1, x_2 Damage levels from the tail of the probability distribution of damages

x Realization of X (outcome, damage)

X Random variable

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